$$\mathfrak{sl}_{2,\mathbb{Q}} := \mathbb{Q}\cdot e \oplus \mathbb{Q}\cdot h \oplus \mathbb{Q}\cdot f$$

$$\left.\begin{array}{ll} [e,h] = 2e & [e,f] = h \\ [f,h] = -2f & \end{array}\right\} (*)$$

$V$: $\mathbb{Q}$-v.space, $\rho : \mathfrak{sl}_2 \longrightarrow \mathrm{End}(V)$:

$e, f, h \in \mathrm{End}(V)$ + $(*)$

St = standard rep $= \mathbb{Q}^2$

$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathrm{Sym}^n(\mathrm{St})$ $n+1$ dim'l repr.
irred.

this gives all irreps, dim $< \infty$

$Sym^n(St)$ :

base vectors

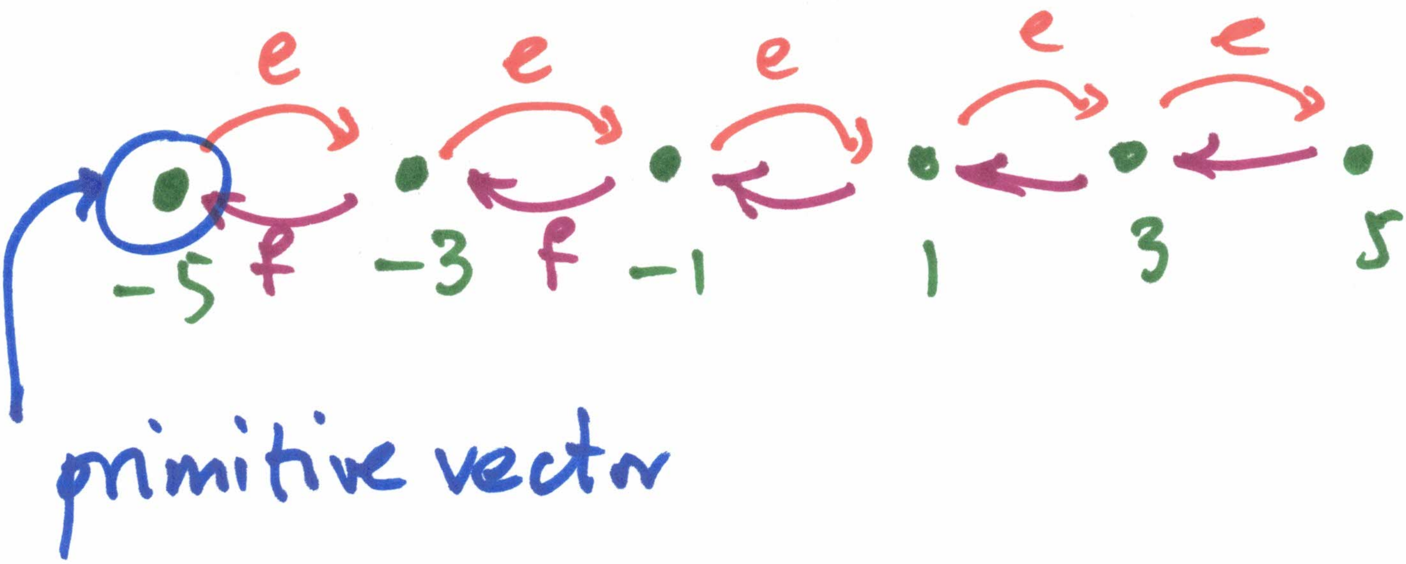
$w_{-n}, w_{-n+2}, \ldots \quad w_{n-2}, w_n$

$h(w_j) = j \cdot w_j$

$e(w_{-n+2i}) = (n-i) \cdot w_{-n+2i+2}$

$f(w_{-n+2i}) = i \cdot w_{-n+2i-2}$

## Exa $n=5$

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$X/k$ AV, $\dim = g$

$\theta : X \longrightarrow X^t$ polarization

$\theta = \varphi_L$ $\quad L$ symmetric ample

$\ell := c_1(L) \in CH^1_{(0)}(X)$

$$\theta \rightsquigarrow \theta_* : CH(X)_{\mathbb{Q}} \xrightarrow{\sim} CH(X^t)_{\mathbb{Q}}$$

$\lambda \in CH^{g-1}_{(0)}(X)$ unique class

s.t. $\theta_*(\lambda) = \mathcal{F}(\ell)$.

<u>Theorem</u>: Op. on $CH(X)_{\mathbb{Q}}$:

$e(\alpha) = \ell \cdot \alpha$ inters. prod

$h(\alpha) = (2i-s-g)\cdot\alpha$ if $\alpha \in CH^i_{(s)}$

$f(\alpha) = \lambda \star \alpha$ Pontr. prod

Then $e, h, f$ satisfy the comm. rels $(*)$ & therefore define a repr. of $\mathfrak{sl}_2$ on $CH(X)_{\mathbb{Q}}$.

For every $s$, the subspace

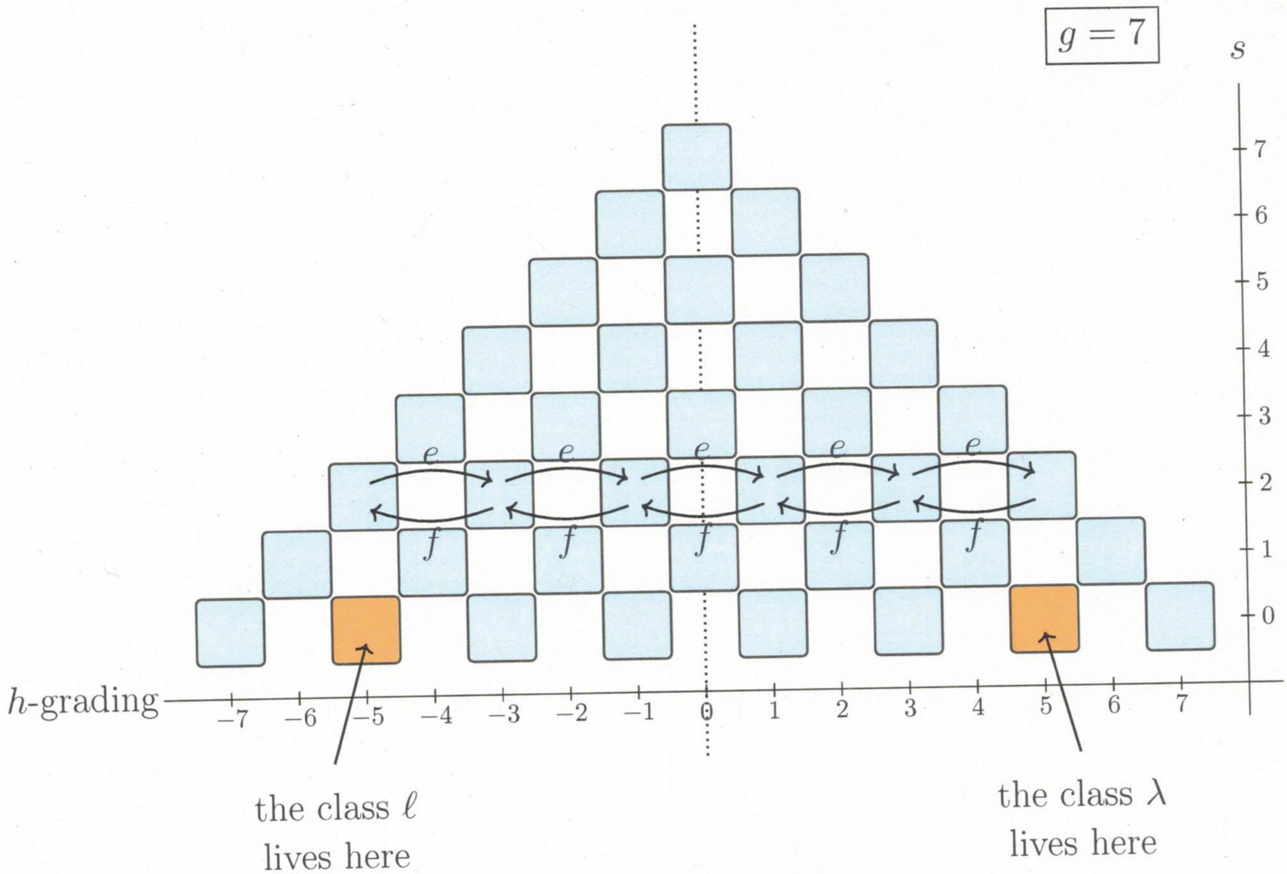
$$CH^{\bullet}_{(s)} := \bigoplus CH^{i}_{(s)}$$

is stable under $\mathfrak{sl}_2$ and:

$\exists$ $\mathbb{Q}$-subspaces $P_{j,s} \subseteq CH^{\bullet\bullet\bullet}_{(s)}$ :

$$CH^{\bullet}_{(s)}(X) = \bigoplus_{\substack{j=0,\dots,g-|s| \\ j \equiv g-s \ (2)}} \underbrace{P_{j,s} \otimes \mathrm{Sym}^{j}(St)}$$

$$e(x) = \ell \cdot x \qquad \text{(intersection product with the class } \ell)$$
$$h(x) = (2i - s - g) \cdot x \qquad \text{if } x \in \mathrm{CH}^i_{(s)}(X)$$
$$f(x) = \lambda \star x \qquad \text{(Pontryagin product with the class } \ell)$$



# 0-cycles

/ $k = \bar{k}$, work <u>integrally</u>

$CH_0(X)$ : ring for $\star$-prod.

$$\deg : CH_0(X) \rightleftarrows \mathbb{Z}$$

$$\sum m_i (P_i) \longmapsto \sum m_i$$

$$m.(0) \longleftarrow\!\!\shortmid \; m$$

$I :=$ $\mathrm{Ker}(\deg)$ $\star$-ideal

$\hookrightarrow$ as $\mathbb{Z}$-mod gen'd by all $(P) - (0)$.

$$CH_0 \supset I \supset I^{\star 2} \supset I^{\star 3} \supset \dots$$

Summation map $S : I \longrightarrow X(k)$ :

$$\sum m_i (P_i) \longmapsto \underbrace{\sum m_i P_i}_{\text{use group law in } X}.$$

<u>Prop.</u> We have s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & I^{\star 2} \arrow[r] & I \arrow[r, "S"] & X(k) \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

$I^{\star 2}$ gen'd by all $((P)-(O)) \star ((Q)-(O))$

$$= (P+Q) - (P) - (Q) + (O)$$

clear: $I^{\star 2} \subseteq \mathrm{Ker}(S)$.

$$((P)-(0)) + ((Q)-(0))$$

$$\equiv ((P+Q)-(0)) \mod I^{*2}$$

$\rightsquigarrow$ every class in $I/I^{*2}$ is repr.'d by some $((P)-(0))$.

$\Rightarrow I^{*2} = \mathrm{Ker}(S)$. $\boxtimes$

Rk: $I = \{\alpha \in CH_0 \mid \alpha \underset{alg}{\sim} 0\}$

$\Rightarrow I$ is divisible

$\Rightarrow I^{*}$ is divisible

<u>Thm</u> (Roitman)

summ. map $S$ induces

$$I_{tors} \xrightarrow{\sim} X(k)_{tors}$$

<u>Cor</u>: $I^{*2}$ torsion free

+ divisible

hence it is a $\mathbb{Q}$-vector space!

$$CH_0(X) \supset I^2 \supset I^{*2} \supset \cdots$$

$$CH_0(X)_{\mathbb{Q}} \supset I_{\mathbb{Q}} \supset I^{*2} \supset \cdots \qquad (A)$$

also:

$$CH_0(X)_{\mathbb{Q}} \supset \bigoplus_{s \geq 1} CH_{0,(s)} \qquad (B)$$

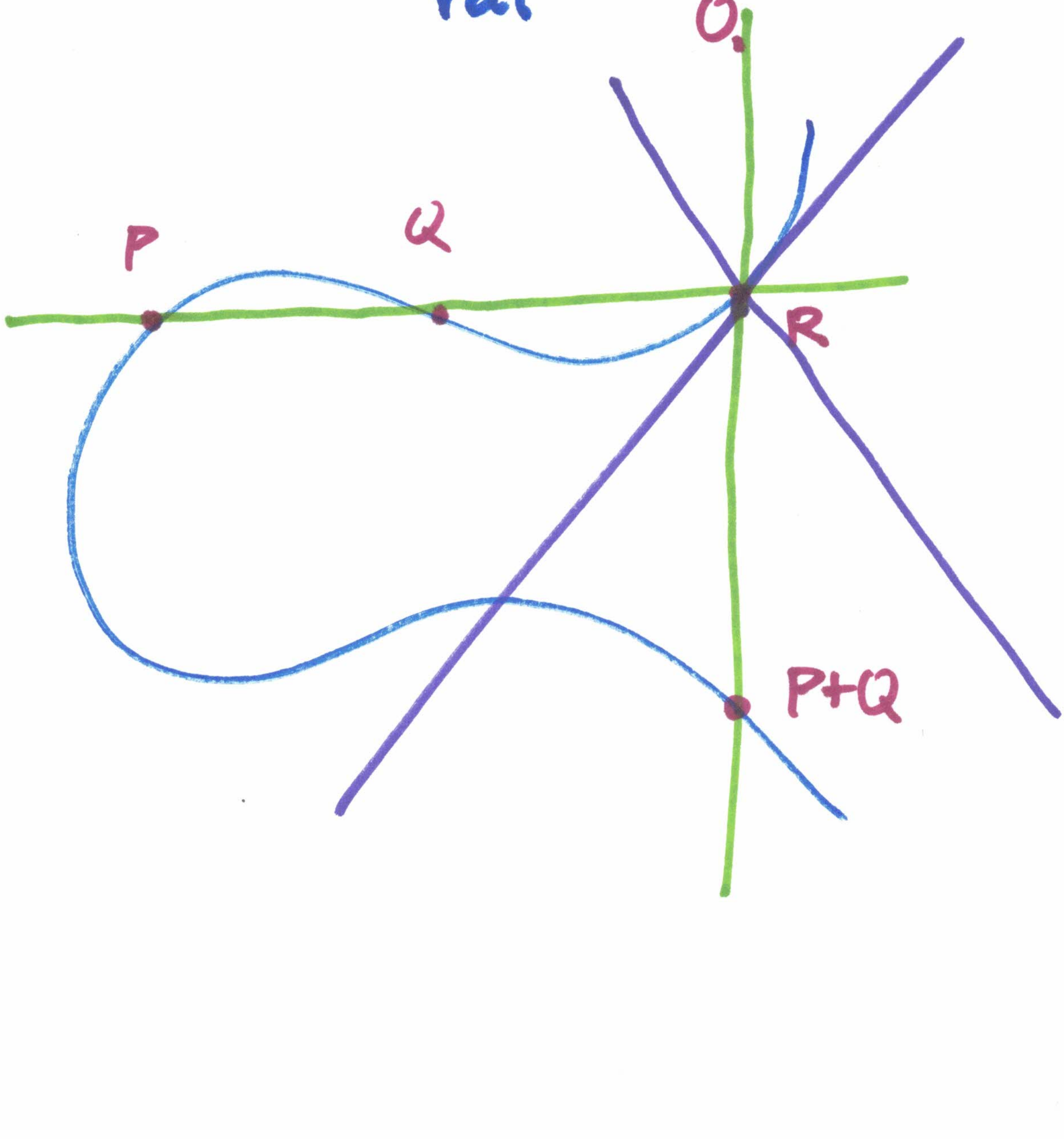
$$\supset \bigoplus_{s \geq 2} CH_{0,(s)} \supset \cdots$$

<u>Theorem</u> (A) and (B) are the same

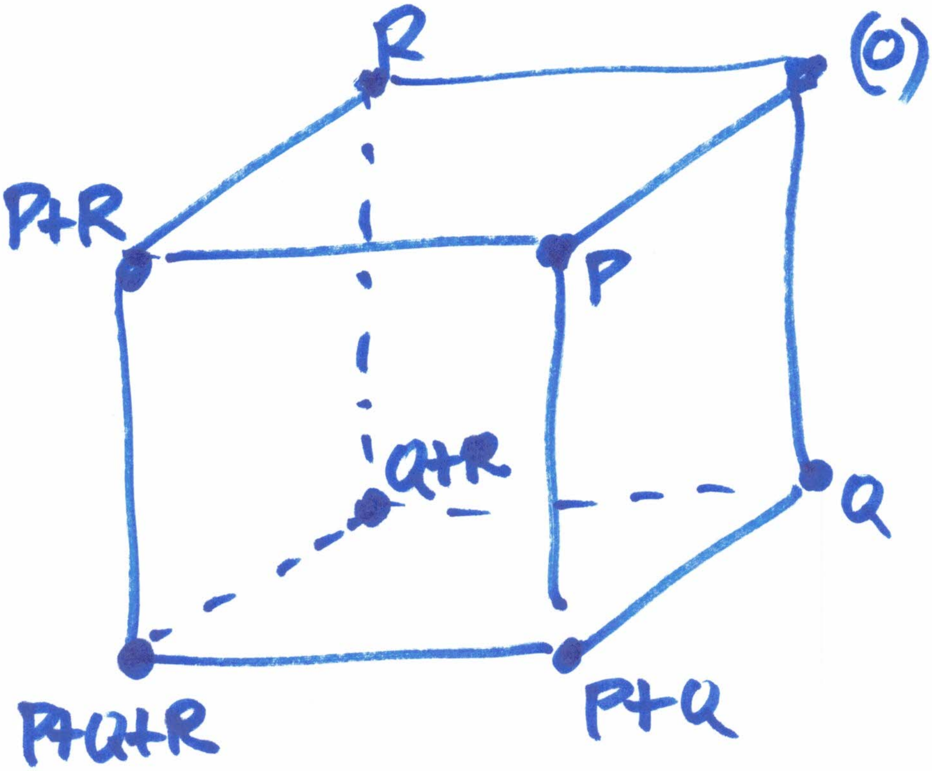
<u>Cor</u>: $I^{*(g+1)} = 0$

**Exa** $g=1$ : $\forall\, P, Q \in X$ :

$$(P+Q) + (O) + (R) \underset{\text{rat}}{\sim} (P) + (Q) + (R)$$



Exa $g=2$ $P, Q, R \in X$




$$(P+Q+R)+(P)+(Q)+(R)$$

$$\sim_{rat} (P+Q)+(P+R) + (Q+R) + (0)$$

General $g$ : $I^{*(g+1)}$ : "hypercube" statement

<u>brief sketch of pf</u> :

- $I = \bigoplus_{s \geq 1} CH_{0,(s)}$

- Beauv. dec. compatible w/ $\star$-prod

  $\Rightarrow \quad I^{\star r} \subseteq \bigoplus_{s \geq r} CH_{0,(s)}$

  (enough for Corollary)

- $[n]_* I \subseteq I \quad \Rightarrow$

  $[n]_* I^{\star r} \subseteq I^{\star r} \quad \forall r$

- $[n]_*$ induces mult. by $n$ on $I/I^{*2} \cong X(k)$

Since

$$(I/I^{*2})^{\otimes r} \twoheadrightarrow I^{*r}/I^{*r+1}$$

get: $[n]_* \circlearrowright I^{*r}/I^{*r+1}$ as mult by $n^r$

- "weight arg" $\Rightarrow$ (A) = (B)

$\boxtimes$