$cl_\ell : \quad CH^m(X) \otimes \mathbb{Q}_\ell \longrightarrow H^{2m}(X_{\bar{k}}, \mathbb{Q}_\ell(m))$

$/\mathbb{C} : \quad cl : \quad CH^m(X)_{\mathbb{Q}} \longrightarrow H^{2m}(X, \mathbb{Q}(m))$

<u>Conj</u>: Should be $\hookrightarrow$ on $CH^{\bullet}_{(0)}$

$\rightsquigarrow$ $CH^{\bullet}_{(0)}$ are $\mathbb{Q}$-v.space of dim $< \infty$

For $CH^1_{(1)}$ and $CH^g_{(1)}$ we see :

- in gen'l not of finite type
- depends on k

Still : "manageable" because points of some alg. var.

<u>Thm</u> (Soulé, Künnemann)

$\subseteq CH^m(X)_{\mathbb{Q}}$

If $k \subseteq \overline{\mathbb{F}}_p$ then $CH^m_{(s)}(X) = 0$

for all $s \neq 0$

<u>Idea of pf</u>

- reduction to $k = \mathbb{F}_q$,

  let $\varphi \in End(X)$ ← Frob. endom

- $\varphi_* =$ mult by q^d on $CH_d(X)$

  $\varphi^* =$ " q^m on $CH^m(X)$

- $P_m :=$ charpol $(\varphi^* \circlearrowright H^m(X))$

  motivic arg. $\Rightarrow$ $P_m(\varphi^*) = 0$

  on $CH^i_{(s)}(X)$ if $2i - s = m$

- q^i is a root of P_m

  Weil conj $\Rightarrow$ $m = 2i$

  $s = 0$

<u>Cor</u>: $X / \overline{\mathbb{F}}_p$ then:

$$CH_0(X) \cong \mathbb{Z} \oplus X(\overline{\mathbb{F}}_p)$$

<u>Reason</u>: $CH_0 \cong \mathbb{Z} \oplus I$

$$\begin{array}{c} 0 \to I^{*2} \to I \xrightarrow{S} X(\overline{\mathbb{F}}_p) \to 0 \\ \parallel \\ \bigoplus_{s \geq 2} CH_{0,(s)} \\ \parallel \\ 0 \end{array}$$

$Y / k = \bar{k}$ sm. proj. var

$CH_0(Y) \supset CH_0(Y)_{hom}$

$\boxed{?}$ What would it mean to say CH_0 is "small"?

- **Property A :**

$\exists\, m$ s.t. every class in $CH_0(Y)_{hom}$ can be repr'd as

$P_1 + \dots + P_m - Q_1 - \dots - Q_m$

- $\gamma_m : Y^m \times Y^m \longrightarrow CH_0(Y)_{hom}$

$(P_1, \dots, P_m, Q_1, \dots, Q_m) \longmapsto P_1 + \dots + P_m - (Q_1 + \dots Q_m)$

fibres are countable unions of alg. subvars.

$d_m = 2m \cdot \dim(Y) - \big($max. dim of a subvar contained in a fibre$\big)$

Property B

$m \longmapsto d_m$ bounded

Property C :

$\exists$ nonsing curve C + $j : C \longrightarrow Y$

such that

$$j_* : CH_0(C)_{hom} \twoheadrightarrow CH_0(Y)_{hom}$$

Prop. D Choose $y_0 \in Y(k)$

$$alb : CH_0(Y)_{hom} \longrightarrow Alb_Y(k)$$

is an $\cong$

Thm (A) – (D) are all equiv.

Theorem (Mumford '69 ~ Roitman)

Suppose $Y/\mathbb{C}$ is sm. proj. such that (A) – (D) hold.

Then $H^0(Y, \Omega^i_Y) = 0 \quad \forall i \geq 2$

Cor $X/\mathbb{C}$ AV, $g \geq 2$

then (A) – (D) do not hold.

Thm (Bloch) $k = \bar{k}$, char $= 0$

uncountable

X/k dim $= g$

$I = CH_0(X)_{hom}$

$I^{*g} \neq 0$ (recall: $I^{*(g+1)} = 0$)

Eg: All $CH_{0,(s)}$ $s = 0, \ldots, g$

are nonzero

False in char $= p$!

X supersing then all $CH^i_{(s)}$

$s > 2$ vanish

"complexity" of CH :

- Kimura / O'Sullivan :

fin. dim'l Chow motives

Know "motives of ab. type"

is fin. dim

- Voevodsky's smash nilp. conj :

$\alpha \in CH^i(Y)$ if $\alpha \underset{num}{\equiv} 0$ $(\Leftarrow \alpha \underset{hom}{\sim} 0)$

then $\exists N$:

$$\alpha^{\otimes N} : \underbrace{\alpha \times \alpha \times \cdots \times \alpha}_{N} \in CH(Y^N)$$

$$\alpha^{\otimes N} = 0$$

Known if $\alpha \overset{alg}{\sim} 0$

Known for 1-cycles on AV